SOME EXAMPLES IN PI RING THEORY†

BY

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ABSTRACT

Several examples are constructed, including a finite ring which cannot be embedded in matrices over any commutative ring, a semiprime PI ring with no classical ring of quotients, an example showing that the property of having all regular elements invertible is not inherited by matrix rings $M_n(R)$, and a prime PI ring R with an idempotent e such that R/ReR has finitely generated projective modules not induced by any finitely-generated projective R-module.

Introduction

The three parts of this paper are independent.

In Part I we develop some necessary conditions for a ring R to be embeddable in matrices over commutative rings (and also in various related sorts of rings, for example, direct products of quasilocal rings), and find that for p a prime the finite ring $\mathrm{End}(\mathbf{Z}_p \oplus \mathbf{Z}_{p^2})$ is not so embeddable.

In Part II, we construct several rings by a method that gives us close control of zero-divisors. In particular, we obtain : (a) a very simple example of a commutative ring with a finitely generated ideal H such that $\mathrm{Ann}\, H = \{0\}$, but every member of H has nonzero annihilator; (b) a semiprime PI ring having no classical (right or left) ring of quotients, and (c) a semiprime PI ring in which every regular element is invertible, but whose $n \times n$ matrix ring does not have this property. The proofs of (b) and (c) use some interesting lemmas on determinants of generic matrices.

In Part III we describe a PI ring R having an idempotent element e such that

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R/ReR has a finitely generated projective module not induced by any finitely generated R-module. The proof uses algebraic topology.

Each of these results suggests further questions and conjectures.

I. A finite ring not embeddable in matrices

We begin by proving some necessary conditions for embeddability of a ring in matrices over a commutative ring.

We shall call a ring *semilocal* if its quotient by its Jacobson radical is semisimple artinian, *quasilocal* if this quotient is simple artinian.† In particular a matrix ring over a local ring is quasilocal.

LEMMA 1. *Let k be a commutative ring and R a k-algebra. Then the following implications hold among the conditions listed below:*

$$\begin{array}{ccccc} & \nearrow & (2) & \searrow & \\ (1) & & & & (3) \rightarrow (4') \leftrightarrow (4). \\ & \searrow & (1') & \nearrow & \end{array}$$

(*Here* embeddable *will mean* as a k-algebra.)

(1) *R is embeddable in a matrix ring over some commutative k-algebra A.*

(1′) *R is embeddable in the endomorphism ring of some (not necessarily finitely generated) projective module over a commutative k-algebra A.*

(2) *R is embeddable in a direct product of quasilocal k-algebras.*

(3) *R is embeddable in a direct product ΠS_λ of k-algebras S_λ with the property that every nonzero finitely generated projective right S_λ-module is faithful (as an S_λ-module).*

(4) *For any idempotent element $e \in R$, and any element $c \in k$,*

$$ce = 0 \Rightarrow ReR \cap cR = \{0\}.$$

(4′) *R is embeddable in a direct product of k-algebras satisfying* (4).

PROOF. (1) $\Rightarrow$(1′) is trivial. (1) implies (2) because any commutative ring is embeddable in a direct product of (commutative) local rings. This same fact,

† In the latter usage I am following M. Ramras. However, commutative ring theorists use "quasilocal" to mean "local, not necessarily Noetherian." What I call quasilocal P. M. Cohn calls "matrix-local". But not every such ring is a matrix ring over a local ring.

together with the fact that a projective module over a local ring is free, hence if nonzero, faithful, gives (1′) ⇒ (3).

To show (2) ⇒ (3) we will show that a quasilocal ring S has the property indicated in (3). Write $S/J(S) = M_n(D)$, D a division ring. If P is a finitely generated projective right R-module, then by the theory of modules over matrix rings, $(P/PJ(S))^n$ will be free over $S/J(S)$, say of rank m. Then P^n and S^m are projective S-modules that become isomorphic modulo $J(S)$, hence are isomorphic over S, hence P^n is free, hence P, if nonzero, is faithful. (The same result for non-finitely-generated S-modules can be gotten by the method of [2], incidentally.)

To see (3) ⇒ (4′) it will suffice to see that an algebra over which nonzero finitely generated projective modules are faithful satisfies (4). But if R is such an algebra and $ce = 0$, then either $e = 0$, or the projective module eR is faithful, so c must be zero in R. In either case, $ReR \cap cR = \{0\}$.

(4) ⇒ (4′) is trivial, and (4′) ⇒ (4) straightforward. ∎

COROLLARY 2. *Let R be a k-algebra, e an idempotent element of R, and c an element of k such that $ce = 0$. Then $ReR \cap cR$ is contained in the kernels of al homomorphisms of R into matrices over commutative k-algebras, into quasilocal k-algebras, and into k-algebras with all nonzero projective modules faithful.* ∎

THEOREM 3. *Let p be any prime (2, for example) and R the endomorphism ring of the abelian group of order p^3, $\mathbf{Z}_p \times \mathbf{Z}_{p^2}$ R has p^5 elements and is semilocal, but cannot be embedded in matrices over any commutative ring.*

In fact, pR will be in the kernel of all homomorphisms of R into commutative rings, quasilocal rings, etc. (as above).

PROOF. Let $e_1, e_2 \in R$ denote the projections of $\mathbf{Z}_p \times \mathbf{Z}_{p^2}$ onto its first and second factors respectively; let f_{21} denote the natural embedding map of $\mathbf{Z}_p$ into $\mathbf{Z}_{p^2}$ (a generator of Hom $(\mathbf{Z}_p, \mathbf{Z}_{p^2})$), and f_{12} the quotient map, $\mathbf{Z}_{p^2} \Rightarrow \mathbf{Z}_p$. Then as an additive group, $R = (e_1) \oplus (f_{12}) \oplus (f_{21}) \oplus (e_2)$. These summands have orders p, p, p and p^2, so R has p^5 elements.

Now $pR = (pe_2) = (f_{21}e_1f_{12}) = Re_1R \cap pR$.

Since $pe_1 = 0$, the above intersection will lie in the kernels of the indicated types of maps.

R is semilocal because it is finite. $(R/J(R) \cong Z_p \times Z_p)$. ∎

The fact that pR is in the kernel of all homomorphisms into quasilocal rings shows that quasilocalization cannot play as nice a role in noncommutative (even

PI) ring theory as localization does in commutative ring theory. (This is also suggested in a different way by [5, Cor. 6.9]: the class of prime ideals of a prime PI ring at which one might quasilocalize is too limited. On the other hand, [5, Cor. 4.2], together with Procesi's central localization, suggest that *semilocal* PI rings are abundant enough to take the part of the local rings of the commutative theory).

The category-minded reader may like the following version, which is not hard to deduce:

COROLLARY 4. *Let p be a prime, and C the full subcategory of* **Ab** *with the two objects $\mathbf{Z}_p$ and $\mathbf{Z}_{p^2}$. Then C cannot be mapped by a faithful additive functor into the category of projective modules over any commutative ring R, or more generally, over a ring R having any of the properties of Lemma 1 (with $k = \mathbf{Z}$). Specifically, any additive functor into such a category will annihilate $p \cdot Id(\mathbf{Z}_{p^2})$.* ∎

Note that if in the construction of the ring R of Theorem 2, we had replaced the base ring of integers by a polynomial ring $F[x]$ (F a field), and p by x, we would have obtained an $F[x]$-algebra 5-dimensional over F, but not embeddable *as an $F[x]$-algebra* in any matrix ring over a commutative $F[x]$-algebra. It is, of course, embeddable *as an F-algebra* in the 3×3 matrix ring over F.

Curiously, however, though this R satisfies the identities of 2×2 matrices over commutative rings (see below) it cannot be embedded in such a 2×2 matrix ring. To show this it suffices to prove that any homomorphism $r \to \bar{r}$ of R into 2×2 matrices over a commutative local ring A will kill xe_2. Given such a map, note that the idempotent 2×2 matrices $\bar{e}_1$ and $\bar{e}_2 = 1 - \bar{e}_1$ will have for images projective, hence free A-modules, say of ranks r_1 and r_2 respectively. The pair (r_1, r_2) must be either $(0,2)$, $(1,1)$ or $(2,0)$. In the first case $\bar{e}_2 = 0$ and in the last $\bar{e}_1 = 0$, so in either case, $\overline{xe_2} = \bar{f}_{21}\bar{e}_1\bar{f}_{12} = 0$. In the middle case we can take coordinates so that $\bar{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\bar{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $\bar{f}_{12}$ and $\bar{f}_{21}$ take the forms $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. So $0 = \bar{f}_{12}\bar{f}_{21} = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix}$, while $\overline{xe_2} = \bar{f}_{21}\bar{f}_{12} = \begin{pmatrix} 0 & 0 \\ 0 & ba \end{pmatrix}$. By commutativity of A, $ba = ab = 0$, hence $\overline{xe_2} = 0$.

Our results leave open the following question.

QUESTION 5. Let F be a field, A a commutative F-algebra, and M a finitely generated A-module, say on n generators. Note that $R = \mathrm{End}(M)$ will be a quotient of a subring of the $n \times n$ matrices over A, hence will satisfy all the identities of $n \times n$ matrices over commutative F-algebras. Can R be embedded as an F-algebra in $n' \times n'$ matrices over some commutative F-algebra A', for some n'? (The preceding example shows that we cannot generally take $n' = n$.) If the answer to the question as it stands is no, it might become true if A is required to be noetherian, of finitely-generated.

Lance W. Small [9] has found an example of an algebra over a field, which is finitely generated, and satisfies the identities of 2×2 matrices over commutative rings, but is not embeddable in matrices over any commutative ring. (This improves on an earlier example of his [8].)

II. Zero-divisors at large

A. A finitely generated ideal of zero-divisors with zero annihilator. Let F be an algebraically closed field, and let $R = F[X, Y]$ be the commutative polynomial ring over F in two indeterminates. We may think of R as the algebra of F-valued functions on the plane F^2, generated by the two coordinate functions X and Y. Clearly, if we delete the origin, R is still faithfully represented on the punctured plane $S = \{(x, y) \in F^2 \mid (x, y) \neq (0,0)\}$.

Now let I denote the space of F-valued functions on S with *finite support*, and let R' denote the function ring $R + I$; that is, the functions on S agreeing almost everywhere with members of R. An element of R' will be a zero-divisor in R' if and only if it is zero at some point of S. "Only if" is clear. Conversely, if a function is zero at $s \in S$, it will annihilate the characteristic function of $\{s\}$.

Note that the elements X and Y have no common zero on the punctured plane S; hence they have no common annihilator in R'. On the other hand, since F is algebraically closed, every element of the ideal $XR + YR$ of R has infinitely many zeroes on S; hence the same is true of every element of $XR' + YR'$, since each such element agrees almost everywhere with an element of $XR + YR$. Hence the ideal $XR' + YR'$ has no regular elements.

B. A semiprime PI ring without a classical ring of quotients. In the preceding example we took a polynomial ring in two variables X and Y, considered it as a function ring, and threw in enough zero divisors so that everything but the ideal (X, Y) had annihilators. In the next example we shall similarly take a ring of generic matrices in two indeterminates X and Y, consider it as a ring of matrix-valued functions, and throw in enough zero-divisors so that essentially everything but powers of Y have annihilators. We shall see that this makes it impossible to write $XA = YB$ with A regular; so the regular elements of R' do not satisfy the right Ore condition, and one cannot construct a classical right quotient ring for R. But we need some algebraic preparation:

1. *Generic matrices and their determinants.* Let F be a field, d and r positive integers, and $R = F\langle X^{(1)}, \cdots, X^{(r)}\rangle$ the $d \times d$ generic matrix algebra in r indeterminates over F. That is, form a commutative polynomial ring in d^2r indeterminates, $C = F[X_{ij}^{(1)}, \cdots, X_{ij}^{(r)} \mid 1 \leqq i, j \leqq d]$, and let R be the subalgebra of the full $d \times d$ matrix ring $M_d(C)$ generated by the r matrices

$$X^{(1)} = ((X_{ij}^{(1)})), \ \cdots, \ X^{(r)} = ((X_{ij}^{(r)})).$$

LEMMA 1. *(Idea from Amitsur [1, Th. 17]). If $A \in R$ and* $\det A = 0$ *in C, then $A = 0$ in R.*

PROOF. Amitsur shows in [1, Lem. 15] that one can construct a division algebra D over F which has dimension d^2 over its center. D will be embeddable in the matrix ring $M_d(K)$ for some field extension K of F, and the polynomial identities in r variables satisfied in D will be precisely those satisfied in all $d \times d$ matrix rings over commutative F-algebras.

Now if $\det A = 0 \in C$, then for every family of r elements $\xi_1, \cdots, \xi_r \in D \subseteq M_d(K)$, we will have $\det A(\xi_1, \cdots, \xi_r) = 0 \in K$, and so the element $A(\xi_1, \cdots, \xi_r) \in D \subseteq M_d(K)$ will be noninvertible. But a noninvertible element of a division ring must be 0. So $A = 0$ is a polynomial identity for D, and hence a relation holding in the generic matrix ring R. ■

Note next that the rings $R = F\langle X^{(1)}, \cdots, X^{(r)}\rangle$ and $C = F[X_{ij}^{(1)}, \cdots, X_{ij}^{(r)}]$ have natural $\mathbf{Z}^r$-gradings. Suppose "$\leqq$" is any total ordering on $\mathbf{Z}^r$ as an additive group. Then we can define the degree in $\mathbf{Z}^r$ of a nonzero element of R or C as the degree of its nonzero homogeneous component maximal under this ordering. From the above lemma and the multilinearity of the determinant, we see:

COROLLARY 2. *If $A \in R - \{0\}$, then for any ordering "$\leqq$" as above,*

$$\deg(\det A) = d \cdot \deg(A). \quad ■$$

PROPOSITION 3. *Suppose $A \in R$, and H is a subset of $\{1, \cdots, r\}$ such that* $\det A$ *lies in the polynomial subring $F[X_{ij}^{(h)} \mid h \in H] \subseteq C$. Then A lies in the subring $F\langle X^{(h)} \mid h \in H\rangle \subseteq R$.*

PROOF. Let $\mathbf{Z}^H \subseteq \mathbf{Z}^r$ denote the subgroup spanned by the basis elements indexed by elements of H. Then we can choose a total ordering of $\mathbf{Z}^r$ such that each of the other basis elements are larger than all elements of $\mathbf{Z}^H$. Then we find $A \in F\langle X^{(h)} \mid H\rangle \Leftrightarrow$ all $\mathbf{Z}^r$-homogeneous components of A have degree in $\mathbf{Z}^H \Leftrightarrow \deg(A) \in \mathbf{Z}^H \Leftrightarrow \deg(\det(A)) \in \mathbf{Z}^H \Leftrightarrow$ all components of $\det(A)$ have degree in $\mathbf{Z}^H \Leftrightarrow \det(A) \in F[X^{(h)} \mid H]$. ■

2. *R and C as function algebras.* Let us now take $r = 2$ and write X and Y for $X^{(1)}$, $X^{(2)}$. Thus $C = F[X_{ij}, Y_{ij}]$ and $R = F\langle X, Y\rangle$. From Proposition 3 we easily deduce:

COROLLARY 4. *If $A \in R$, and* $\det A$ *is a scalar multiple of a power of* $\det Y$, *then A is a scalar multiple of a power of Y.* ■

Let us also assume F algebraically closed, and let us represent C as the F-valued function algebra on the space $F^{2d^2} = \{(x_{11}, \cdots, x_{dd}, y_{11}, \cdots, y_{dd}) \mid x_{ij}, y_{ij} \in F\}$ generated by the $2d^2$ coordinate functions X_{ij} and Y_{ij}. Then R can be considered an algebra of $d \times d$ matrix-valued functions on this space. In fact, if we write F^{2d^2} as $M_d(F)^2 = \{(x, y) \mid x, y \in M_d(F)\}$, the generators X and Y of R become the two coordinate functions on this space.

We shall say that an element of R is singular *at* a point of F^{2d^2} if its value at that point is a singular matrix.

COROLLARY 5. *Suppose $A \in R$ is not a scalar multiple of a power of Y. Then A is singular at some point of F^{2d^2} at which Y is nonsingular — in fact, at infinitely many such points.*

PROOF. If A is not a scalar multiple of a power of Y, then by Corollary 4, $\det A$ is not a scalar multiple of a power of $\det Y$. Since the polynomial ring C is a unique factorization domain, and $\det Y$ is irreducible, this means $\det A$ has some irreducible factor other than $\det Y$, and the conclusion follows by the Nullstellensatz applied to C. ■

We shall combine the above corollary with:

LEMMA 6. *If $d > 1$, then no equation of the form*

$$XY^n = YB \qquad (n \geqq 0) \tag{1}$$

holds in R.

PROOF. Assume the contrary. Write B as a polynomial $B(X, Y)$. Clearly this polynomial may be taken[†] homogeneous, of degree 1 in X, hence every monomial occurring in YB will involve a factor YX.

But it is easy to choose matrices x and y in $M_d(F)$ such that y is idempotent and $yx = 0$, but $xy \neq 0$. (For example $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, extended by 0's to $d \times d$ matrices.) Substituting these for X and Y in (1), the left side becomes nonzero and the right side zero, contradiction. ∎

Corollary 7. *If $d > 1$, and A, B are elements of R such that $XA = YB$, then there exist infinitely many points of $M_d(F)^2$ at which Y is nonsingular but A is singular.* ∎

3. *The example.* In $M_d(F)^2 = F^{2d^2}$, let V_Y denote $\{(x, y) \mid \det y = 0\}$. This is a proper subset, and closed in the Zariski topology on F^{2d^2}, so its complement is dense, hence C and R are faithfully represented on this complement, $S = M_d(F)^2 - V_Y$. We shall henceforth consider C and R as rings of functions on S.

Let I denote the vector-space of $M_d(F)$-valued functions on S with finite support, and R' the ring of functions generated by R and I. As in Part A, we see that I will be an ideal of R', and in fact, $R' = R \oplus I$ as vector spaces. The projection $R' = R \oplus I \to R$ is clearly a ring homomorphism. Note that R' is a subdirect product of copies of $M_d(F)$, hence a semiprime ring with polynomial identity.

An element $f \in R'$ will be (right, equivalently left) regular in R' if and only if it is nonsingular at every point of S. "If" is clear. Conversely, if f is singular at $s \in S$, it will be right (left) annihilated by an element with support $\{s\}$ and value at s chosen to right (left) annihilate $f(s)$. In particular, by our choice of S, Y is regular in R'.

Theorem 8. *The semiprime ring with polynomial identity, R', has no classical ring of right (or left) quotients.*

Proof. Suppose A and B are elements of R, and α, β elements of I, such that $X(A + \alpha) = Y(B + \beta)$ in R'. Then $XA = YB$ in R, so by Corollary 7, A is singular at infinitely many points of S, hence so is $A + \alpha$, so this is not regular.

[†] B will be homogeneous as an element of R; but since R is not a free associative algebra, B will have more than one representation as a polynomial, and some of these will be nonhomogeneous. However, a homogeneous element of R always has a representation by a homogeneous polynomial of the same degree.

Thus the right Ore condition is not satisfied. Since the construction of R' is left-right symmetric, the left Ore condition also fails for X and Y. ∎

4. *Regular elements without a regular common right multiple.* Suppose we modify the above construction by taking $S = M_d(F)^2 - V_X - V_Y$, the set of points where X and Y are both nonsingular. Then X and Y will both be regular elements of R'. Can we again prove that the equation $X(A+\alpha) = Y(B+\alpha)$ has no solution with $A+\alpha$ regular, and thus obtain a PI ring in which these two regular elements have no regular common right multiple?

To do this by determinants, as above, we would need the following analog of Corollary 4, which I have not been able to prove:

CONJECTURE 9. *If* $A \in R$, *and* $\det A$ *has the form* $(\det X)^p(\det Y)^q$, *then* A *is equal to a monomial in* X *and* Y *(a scalar times a product of* X's *and* Y's).

However, with the help of some recent results one can obtain a different proof, which I will sketch.

We want to know whether in the generic $d \times d$ matrix algebra $R = F\langle X, Y\rangle$ there exist polynomials $A(X,Y)$ and $B(X,Y)$ such that $XA(X,Y) = YB(X,Y)$, but which are nonsingular at all points of $M_d(F)^2 - V_X - V_Y$. If such A and B exist, let us call them a pair of *d-miraculous* polynomials for F. (When $d = 1$, such polynomials exist: $A = Y$, $B = X$.) If A and B are d-miraculous, they will also be d'-miraculous for all $d' \leqq d$. This can be seen by enlarging arbitrary $d' \times d'$ nonsingular matrices x and y to $d \times d$ nonsingular matrices $\begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}$ and $\begin{pmatrix} y & 0 \\ 0 & v \end{pmatrix}$. Assuming F algebraically closed, it is easy to see that if A and B are d-miraculous for F they will be d-miraculous for any extension field F', and it is easy to deduce that if T is any order in a division algebra over F, of dimension $\leqq d^2$ over its center, then in T, $(x \neq 0,\ y \neq 0) \Rightarrow (A(x,y) \neq 0,\ B(x,y) \neq 0)$.

We shall also need the following lemma, not difficult to prove. (For the definition and a discussion of specializations, see [4, Sect. 2]; also [6, Sect. 7.2]. For Ore right denominator sets see [6, Sect. 0.5].)

LEMMA 10. *Let* T *and* T' *be right Ore domains with skew fields of fractions* E *and* E' *and* $f\colon T \to T'$ *a homomorphism. Then a sufficient condition for* f *to extend to a specialization from* E *to* E' *is that* $T - \ker f$ *be an Ore right denominator set in* T. ∎

(I suspect that this condition is not necessary, however.)

Now let T and T' be F-algebras without zero-divisors satisfying the identities of $d \times d$ matrices; that is, orders in division algebras of central dimension $\leqq d^2$. I claim that if there exists a d-miraculous pair of polynomials A, B for F, then the condition of the above lemma holds for any $f\colon T \to T'$.

Indeed, we see that the set $S = T - \ker f$ will be a right Ore semigroup. If $x, y \in S$, then $xA(x,y) = yB(x,y)$ in S. This semigroup generates the ring T. (For any $x \in T$, either $x \in S$, or $1, x - 1 \in S$). But it is easy to check that if a ring T is generated by a right Ore semigroup S, then S is a right Ore denominator set for T. Hence by Lemma 10, f extends to a specialization of skew fields of fractions.

But now let T be a generic $d \times d$ matrix algebra over F in more than one indeterminates, and T' the corresponding $d' \times d'$ generic matrix algebra, for any $d' \leqq d$. There is a natural homomorphism of T onto T', but it is proved in [4, Th. 4.1] (also, essentially, [5, Cor. 6.9]) that this extends to a specialization of skew fields of fractions if and only if d' divides d. If $d \geqq 3$, then taking $d' = d - 1$, the specialization will not exist; hence for $d \geqq 3$ there can exist *no* d-miraculous polynomials. I do not know about the case $d = 2$.

We can now complete the construction indicated above: Take $d \geqq 3$; represent the generic $d \times d$ matrix ring R as a ring of $M_d(F)$-valued functions on $S = F^{2d^2} - V_X - V_Y$; let I denote the space of all $M_d(F)$-valued functions on S with finite support, and put $R' = R + I$. Then we see:

PROPOSITION 11. *The semisimple* PI *algebra R' described above has two regular elements X and Y with no regular common right multiple.* ■

C. A quo-wrong matrix ring. We shall construct here for any integers $d, n > 1$ a semisimple ring R' of $d \times d$ matrix-valued functions over a field F, such that R' is its own classical ring of quotients (all regular elements of R' are invertible) but $M_n(R')$ fails to have this property.

This answers in the negative a question asked by Faith [7, p. 229, last sentence].

Let F be an algebraically closed field, C be the commutative polynomial ring over F in n^2d^2 indeterminates X_{pq} $(p, q \leqq nd)$, and let R be the subring of $M_d(C)$ generated by the n^2 generic matrices

$$X_{(\iota\eta)} = ((X_{(\iota-1)d+i(\eta-)d+j}))_{i,j \leqq d} \quad (\iota, \eta \leqq n).$$

Regard C and R as rings of F- and $M_d(F)$-valued functions on $F^{d^2n^2}$. Then

$M_n(R) \subseteq M_{nd}(C)$ can be considered as a ring of $M_{nd}(F)$-valued functions on the same space. Within this ring, let X denote the $n \times n$ matrix $((X_{(\iota\eta)}))$, or as a member of $M_{nd}(C)$, the $nd \times nd$ matrix $((X_{pq}))$. Let $V_X \subseteq F^{n^2d^2}$ denote the set of points at which X is singular; that is, the zero-set of $\det X \in C$. We now make the same sort of construction we have before: Let $S = F^{n^2d^2} - V_X$, represent C and R by rings of functions on S, let I be the space of $M_n(F)$-valued functions on S with finite support, and take $R' = R \oplus I$. An element of R' or $M_n(R')$ will be regular if and only if it is nonsingular at every point of S. In particular, $X \in M_n(R')$ is regular; but it is clearly not invertible. (That would imply that it was invertible in $M_n(R)$, and hence that $\det X$ was invertible in C.) Hence $M_n(R')$ is not its own classical ring of quotients.

But now consider an element $A + \alpha$ of $R'(A \in R,\ \alpha \in I)$. If A is a constant function (member of F) then both regularity and invertibility of $A + \alpha$ are easily seen equivalent to the conditions: $A \neq 0$, and $A + \alpha$ is nonsingular on the finite set where $\alpha \neq 0$. In the general case, a necessary condition for regularity is that A be nonsingular throughout S, which turns out to mean, by the same kind of argument we used before. that $\det A \in C$ shoold be a scalar multiple of a powe of $\det X$. (Note that the former determinant is of a $d \times d$ matrix, and the latter of and $nd \times nd$ matrix.)We shall show that no nonscalar $A \in R = F\langle X_{(11)}, \cdots, X_{(nn)}\rangle$ has this property. It will follow that the only regular elements $A + \alpha \in R'$ are those with A scalar, which we have observed are invertible, so that R' is its own classical ring of quotients, as claimed.

Suppose, then, that we had $A(X_{(11)}, \cdots, X_{(nn)}) \in F\langle X_{(11)}, \cdots, X_{(nn)}\rangle$ such tha

$$\det A = c(\det X)^r \quad (c \in F - \{0\}, r \geqq 0). \tag{2}$$

Let us regard the argument of the noncommutative polynomial A, not as a linear sequence of n^2 elements, but as an $n \times n$ matrix of elements of R, or as an $nd \times nd$ matrix of elements of C. Thus A can be thought of as representing a certain function which associates to any $nd \times nd$ matrix ξ over a commutative F-algebra B a $d \times d$ matrix $A(\xi)$ formed by certain polynomial operations from the n^2 $d \times d$ blocks into which ξ decomposes.

Now let us take for B as in the above paragraph a polynomial ring in a number of n^2-tuples of indeterminates over F, $B = F[W_{ij}, Y_{ij}, Z_{ij}, \cdots \mid i,j \leqq d]$, and write $W, Y, Z, \cdots$ for the corresponding generic matrices of $M_d(B)$.

If in (2) we substitute for X the diagonal matrix

$$\tilde{W} = \begin{bmatrix} W & & & & 0 \\ & I & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & I \end{bmatrix}, \tag{3}$$

the result will be a polynomial $P(W) = A(\tilde{W})$ such that $\det P(W) = c(\det W)^r$. Hence $P(W)$ must have the form $c'W^r$ $(c' \in F - \{0\})$. It follows that for any $d \times d$ matrix ω over any commutative F-algebra, we have $A(\tilde{\omega}) = c'\omega^r$.

On the other hand, suppose we substitute for X any product $S(Y, Z, \cdots)$ of finitely many block-elementary matrices in $M_{nd}(B)$, for example,

$$\begin{bmatrix} I & Y \cdot \cdot \cdot & 0 \\ 0 & I & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 \cdot \cdot \cdot & I \end{bmatrix}, \quad \begin{bmatrix} I & 0 \cdot \cdot \cdot & 0 \\ 0 & I & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ Z & 0 \cdot \cdot \cdot & I \end{bmatrix}, \text{ etc.}$$

These matrices all have determinant 1, hence so does the product. Hence by (2), $\det A(S(Y, Z, \cdots)) = c' \in F$; hence by Proposition 3, $A(S(Y, Z, \cdots))$ must be a a scalar $c'' \in F$. It follows that if σ is any matrix over F which can be written as a product of block-elementary matrices, then $A(\sigma) = c''I$.

But I claim that the block-elementary matrices generate all of $SL(nd, F)$. Indeed, it is well known that this is generated by the ordinary elementary matrices

$$I + \mu e_{pq} \quad (\mu \in F,\ p \neq q).$$

Writing $p = (\iota - 1)d + i$, $q = (\eta - 1)d + j$, we see that this will be "block-elementary" if $\iota \neq \eta$. If $\iota = \eta$, it can still be written as a product of block-elementary matrices: for any $q' = (\iota' - 1)d + j'$ with $\iota' \neq \iota = \eta$ (so in particular, $q' \neq p, q$) we have:

$$I + \mu e_{pq} = (I - e_{q'q})(I + \mu e_{pq'})(I + e_{q'q})(I - \mu e_{pq'}),$$

a product of block-elementary matrices.

So now given any $\omega \in SL(d, F)$, we have $\tilde{\omega}$ (as in (3)) $\in SL(nd, F)$, so $A(\tilde{\omega}) = c''I$; but we also showed that $A(\tilde{\omega}) = c'\omega^r$. So $r = 0$, so $A \in F$, as claimed. Thus we have proved:

PROPOSITION 12. *If F is a field and d, n are integers > 1, then there is* no *polynomial A in n^2 generic $d \times d$ matrices $X_{(\iota\eta)}$ such that, writing X for the $nd \times nd$ matrix formed from these blocks, one has*

$$\det A\, X_{(\iota\eta)} = c(\det X)^r \quad (\text{for some } c \in F - \{0\},\ r > 0).$$

Equivalently, there is no *polynomial A in the $X_{(\iota\eta)}$ such that for any $nd \times nd$ matrix ξ over the algebraic closure of F, ξ is singular if and only if $A(\xi_{(\iota\eta)})$ is singular, where the $\xi_{(\iota\eta)}$ are the $d \times d$ block-components of ξ.* ∎

This completes proof of:

THEOREM 13. *The semiprime* PI *ring R' described at the beginning of Part* C *is its own classical ring of quotients, but $M_n(R')$ does not have this property.* ∎

Further observations: For $d = 1$ there *is* a polynomial A with the property considered in Proposition 12: the determinant itself. Thus, Proposition 12 says that for $d > 1$ a certain sort of generalized determinant cannot exist. On the other hand, there do exist nonzero polynomial functions $A(\xi_{(\iota\eta)})$ which are singular when (but not only when) ξ is; equivalently, such that $\det A$ is divisible by $\det X$. For instance, let $n = 2$. Because R is a right Ore ring, we can find nonzero U, $V \in R$ such that $X_{(11)}U + X_{(12)}V = 0$. Then $A = X_{(21)}U + X_{(22)}V$ can be shown to have this property.

One might try to prove (or disprove) a stronger statement than Proposition 12:

CONJECTURE 14. *Let F be a field, S a simple infinite noncommutative F-algebra with polynomial identity (say of dimension $d^2 > 1$ over its center) and $n > 1$ an integer. Then there does not exist a noncommutative polynomial A in n^2 indeterminates $X_{(\iota\eta)}$ such that for all $\xi_{(\iota\eta)} \in S$, $A(\xi_{(\iota\eta)})$ is regular in S if and only if $\xi = ((\xi_{(\iota\lambda)}))$ is regular in $M_n(S)$.*

In particular, given F, d, n there is no A such that for all values of ξ in $nd \times nd$ matrices over any field extension of F, $A(\xi_{(\iota\eta)})$ is nonsingular whenever ξ is, and is singular whenever $\operatorname{rank} \xi \leqq (n-1)d$.

To see that the second assertion would follow from the first, assume we had an A satisfying the conditions of the second assertion, let S be a division algebra over F, of dimension d^2 over its center, embed S in a matrix ring $M_d(K)$, and note that any $n \times n$ matrix ξ over S which is singular will have rank $\leqq (n-1)d$ in $M_{nd}(K)$. Thus for this S, A would satisfy the conditions of the first assertion.

Lance Small has pointed out to me that to wrap the subject up completely, one would like to know whether there exist rings R' with the properties of Theorem 13,

(a) for which $M_n(R')$ has a classical ring of quotients, and

(b) for which $M_n(R')$ does *not* have a classical ring of quotients.

My *guess* is that the example of Theorem 13 probably satisfies (b). For example, if $P \neq I$ is a permutation matrix over R', then PX and X probably have no *regular* common right multiple. I don't know how hard this might be to prove, nor whether examples satisfying (a) are also likely to exist.

III. Unliftable idempotents

Let R be a ring, P a finitely generated projective module, and T_P the trace ideal of P. In [3, Sect. 11] we were led to the question of how closely the properties of R/T_P resemble those of R, for example, is it always true that $\operatorname{gl\,dim}((R/T_P) \leqq \operatorname{gl\,dim} R$? The example given here will show that R/T_P can have cyclic projective modules not induced by any finitely generated projective R-module. Our R will be a finitely generated subalgebra of 2×2 matrices over a commutative Q-algebra.

1. *The example.* Let Q denote the field of rational numbers, and C the commutative Q-algebra $Q[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$; that is, the ring of polynomial functions on the unit sphere. Let ρ denote the map of the unit sphere onto itself taking the point (x, y, z) to $(-x, -y, z)$, and let us write $f \mapsto f^\rho$ for the induced automorphism of C. Thus $f^\rho(X, Y, Z) = f(-X, -Y, Z)$.

Let $R = C\langle r\rangle$ denote the extension of C consisting of all expressions $f + gr$ $(f, g \in C)$ with multiplication defined using the laws $rf = f^\rho r$ and $r^2 = 1$. R may be thought of as the ring of operators on the additive group of C, generated by the multiplications by elements of C, and the operator ρ, called r. Defining relations for the algebra R in terms of the generators X, Y, Z and r are:

$$XY = YX, \ YZ = ZY, \ ZX = XZ, \tag{1}$$

$$X^2 + Y^2 + Z^2 = 1, \tag{2}$$

$$Xr = -rX, \ Yr = -rY, \ Zr = rZ, \tag{3}$$

$$r^2 = 1. \tag{4}$$

Now by (4), $e = (1 - r)/2$ is an idempotent element of R. To obtain the structure of R/ReR, adjoin to (1)–(4) the relation $e = 0$, that is, $r = 1$. From (3) we now obtain $X = Y = 0$, and we find our Q-algebra reduced to one generator, Z, and one nontrivial relation:

$$(2') \qquad Z^2 = 1.$$

The algebra so defined is $Q \times Q$; the map $R \to Q \times Q$ takes Z to $(+1, -1)$ and hence more generally

$$(5) \qquad f(X,Y,Z) + g(X,Y,Z)r \mapsto (f(0,0,1) + g(0,0,1), f(0,0,-1) + g(0,0,-1)).$$

The projective ideals of $Q \times Q$ generated by the idempotents $(1,0)$ and $(0,1)$ will be denoted P_+ and P_- respectively. Every finitely generated $Q \times Q$ module is projective, isomorphic to $P_+^a \oplus P_-^b$ for unique nonnegative integers a and b. We shall determine below which such modules are induced by finitely generated projective R-modules.

2. *A matrix representation.* It is easy to check that R is embedded in the matrix ring $M_2(C)$ via

$$(6) \qquad X \mapsto \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \ Y \mapsto \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix}, \ Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}, \ r \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we $\mathbf{Z}_2$-grade C by defining $C_{(0)} = \{f \in C \mid f^\rho = f\}$, $C_{(1)} = \{f \in C \mid f^\rho = -f\}$, then $R \subseteq M_2(C)$ is easily seen to be the subring

$$((6') \qquad R = \begin{pmatrix} C_{(0)} & C_{(1)} \\ C_{(1)} & C_{(0)} \end{pmatrix}.$$

The idempotent e takes the form $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We may think of each element $A \in R \subseteq M_2(C)$ as a continuous real 2×2 matrix valued function on the unit sphere S^2. Then (6′) tells us that for every $s \in S^2$,

$$7) \qquad \text{if } A(s) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ then } A(s^\rho) = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}.$$

In particular, A must have diagonal values at each of the poles $(0,0,\pm 1)$. Hence the map

$$(5') \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a(0,0,1), a(0,0,-1))$$

is a homomorphism of R into $Q \times Q$; this is clearly the map (5): $R \to R/ReR \cong Q \times Q$!

(To motivate the matrix representation (6), (6′), think of R as a ring of additive endomorphisms of C. These will in fact be $C_{(0)}$-module endomorphisms. Writing

$C = C_{(0)} \oplus C_{(1)}$ as a $C_{(0)}$-module, we may represent each of these endomorphisms by a 2×2 matrix.)

3. *Idempotent elements of* R. We shall show in this section that the idempotent element $(1,0) \in Q \times Q$ is not the image of any idempotent element of R. This will allow us to conclude that the projective $Q \times Q$-modules P_+ and P_- are not induced by any *cyclic* projective R-modules. To determine whether these modules are induced by any *finitely generated* projective R-modules, however, we shall have to study arbitrary idempotent matrices over R. This section, therefore, is really a warm-up for the next section, where we shall obtain a stronger result.

Suppose that A is an idempotent element of R. Thinking of A as a matrix-valued function, this means that the value of A at each point of S^2 is an idempotent matrix, that is, a projection of the real plane onto some subspace. By continuity, the ranks of these subspaces will be constant over S^2, hence if A is not one of the trivial idempotents 0 or 1, it will associate to each point $s \in S^2$ a projection of the real plane onto a *line* through the origin, $L(s)$. We note that by (7), $L(s^\rho) = L(s)'$, where the prime denotes reflection about the X-axis. In particular, at each pole of S^2, $L(s)$ will be either the X-axis or the Y-axis. In fact, at each of these points the value of A will be a diagonal idempotent matrix of rank 1, hence either $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Can $L(s)$ be the X-axis at one pole of S^2, and the Y-axis at the other? Let s move along any path ϕ in S^2 from $(0,0,1)$ to $(0,0,-1)$, and let w denote the total winding number through which the line $L(s)$ turns. $4w$ will be an integer because $L(s)$ starts and ends at an axis. At the same time, the point s^ρ will travel the path ϕ^ρ between the same two points, and the associated line $L(s^\rho) = L(s)'$ will clearly turn through a winding number of $-w$. But the paths ϕ and ϕ^ρ are homotopic because S^2 is simply connected, hence $w = -w$, so $w = 0$, so $L(s)$ must return at $(0,0,-1)$ to the same value it had at $(0,0,1)$. Therefore A is either $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ at both poles, or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ at both poles. Hence we see that the image of A under the map (5′) will be either $(1,1)$ or $(0,0)$. This establishes that $(1,0)$ is not the image of an idempotent element of R.

4. *Idempotent matrices over* R. An $n \times n$ matrix A over R can be thought of as a $2n \times 2n$ real matrix-valued function on S^2. Let us specifically represent A,

not by the $n \times n$ array of 2×2 blocks which are its components in R, but by a 2×2 array of $n \times n$ blocks, where the upper-left-hand block represents the $(1,1)$ entries of all components of A, etc. Then we see that for each $s \in S^2$,

$$(7') \qquad \text{if } A(s) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ (\alpha, \beta, \gamma, \delta \in M_n(R)) \text{ then } A(s^\rho) = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}.$$

Let us consider these matrices to act on the vector space $W = \mathbf{R}^{2n} = U \oplus V = \mathbf{R}^n \oplus \mathbf{R}^n$. That is, we write U for the subspace of $W = \mathbf{R}^{2n}$ spanned by the first n basis vectors and V for the subspaced spanned by the remaining n. Let $(\ )'$ denote the involution of W taking a vector $u + v (u \in U, v \in V)$ to $u - v$. Thus, the matrix transformation in (7′) represents conjugation by this involution.

Suppose A is an idempotent element of $M_n(R)$, that is, an element whose value $A(s)$ at each $s \in S^2$ is an idempotent endomorphism of W. As before A will have constant rank, say m, which we can assume is neither 0 nor $2n$. Again, the subspace onto which $A(s)$ projects W will be called $L(s)$. This will now be thought of as a point of the Grassmannian manifold $G^m(W)$ of all m-dimensional subspaces of W. Condition (7′) says that $L(s^\rho) = L(s)'$, where $(\ )'$ on $G^m(W)$ means the involution induced in the obvious manner by the involution $(\ )'$ of W.

Note that a subspace of W is invariant under $(\ )'$ if and only if it is the direct sum of a subspace of U and a subspace of V. We shall denote by $G^a(U) \times G^{m-a}(V) \subseteq G^m(W)$ the set of all m-dimensional subspaces of W which decompose into the direct sum of an a-dimensional subspace of U and an $(m-a)$-dimensional subspace of V. Thus by the above remark, the fixed set of $(\ )'$ on $G^m(W)$ is the (disjoint) union $\bigcup_a G^a(U) \times G^{m-a}(V)$. (Here a runs over all values making both a and $m - a$ lie between 0 and n; that is, $0, \cdots, m$ if $0 \leqq m \leqq n$, or $m - n, \cdots, n$ if $n \leqq m \leqq 2n$.) In particular, the map $L\colon S^2 \to G^m(W)$ corresponding to our idempotent $A \in M_n(R)$ will take the two poles of S^2 to points of this set; say one pole to a point of $G^a(U) \times G^{m-a}(V)$, the other to $G^b(U) \times G^{m-b}(V)$. Then the image of A under the map (5′) will be an idempotent $n \times n$ matrix over $\mathbf{Q} \times \mathbf{Q}$ determining the projective module $P_+^a \times P_-^b$.

As before, if we consider the image under L of any path ϕ from one pole to the other in S^2, we get a path ψ from a point of $G^a(U) \times G^{m-a}(V)$ to a point of $G^b(U) \times G^{m-b}(V)$, such that ψ and ψ' are homotopic in $G^m(W)$. For which a and b can such a path ψ exist?

Now $\pi_1(G^m(W)) = \mathbf{Z}_2$ (except for the case $n = 1$, $m = 1$, considered in the preceeding section, where it is $\mathbf{Z}$; we may either exclude this case here, or observe that it can easily be adapted to the agrument to follow, by understanding "homotopy mod 2" for "homotopy", and noting that a, $b \in \{0,1\}$ have the same parity if and only if they are equal). It is easily deduced that to answer the above question, it suffices to choose, for each a and b, one path ψ between one such pair of points, and calculate whether ψ and ψ' are homotopic.

Assume by symmetry that $a \leqq b$, and choose linearly independent elements $p_1, \cdots, p_b \in U$ and $q_1, \cdots, q_{m-a} \in V$. For $0 \leqq t \leqq 1$ define $\psi(t) \in G^m(W)$ to be the subspace of W having for basis the m elements:

$$
\begin{array}{ll}
(1-t)p_1 + tq_1, \cdots, (1-t)p_{b-a} + tq_{b-a} & (b - a \text{ elements moving from } U \text{ to } V) \\
p_{b-a+1}, \cdots, p_b & (a \text{ elements of } U \text{ independent of } t) \\
q_{b-a+1}, \cdots, q_{m-a} & (m-b \text{ elements of } V \text{ independent of } t).
\end{array} \tag{8}
$$

This will be a path of the desired sort. In presenting next the spaces $\psi(t)'$ $(0 \leqq t \leqq 1)$ let us give a basis which at $t = 0$ agrees with the basis used for ψ. Such a basis is clearly

$$
\begin{array}{l}
(1-t)p_1 - tq_1, \cdots, (1-t)p_{b-a} - tq_{b-a} \\
p_{b-a+1}, \cdots, p_b \\
q_{b-a+1}, \cdots, q_{m-a}.
\end{array} \tag{8'}
$$

Note that at $t = 1$, the bases given for $\psi(t)$ and $\psi(t)'$ differ only in the signs of the first $b - a$ basis elements.

Now $G^m(W)$ has for its (2-sheeted) universal covering space the manifold of all oriented m-subspaces of W [10, Sect. 25]. If we lift the above two paths to this space, we see that their $t = 1$ endpoints will agree, that is, be spaces with the same orientation, if and only if the *number*, $b - a$, of basis elements whose sign has been changed is *even*. Hence this is the necessary and sufficient condition for ψ to be homotopic to ψ', and is therefore, by our preceding observations, a *necessary* condition for $P_+^a \oplus P_+^b$ to be induced by a finitely generated projective R-module. In the next section, we shall see that it is also sufficient. Meanwhile, we note that as claimed, P_+ is *not* induced by a finitely generated projective R-module.

5. *Complexification.* Let $\bar{R}$ denote $R[i] = R \otimes_Q Q(i)$, a ring of continuous *complex* 2×2 matrix valued functions on S^2. Then $\bar{R}/\bar{R}e\bar{R} = Q(i) \times Q(i)$. Let $\bar{P}_+$ and $\bar{P}_-$ denote the $Q(i) \times Q(i)$-modules generated by $(1,0)$ and $(0,1)$.

Surprizingly, $\bar{P}_+$ and $\bar{P}_-$ *are* induced by (cyclic) projective R-modules: The matrix $1/2\begin{pmatrix} 1+Z & X-iY \\ X+iY & 1-Z \end{pmatrix} \in R \subseteq M_2(\bar{C})$ is idempotent (since it has trace 1 and determinant 0) and is mapped to (1, 0) under (5′). In X, Y, Z, r-form, this element is $1/2((1+X)+(Z+iY)r)$.

If we restrict scalars to $R \subseteq \bar{R}$ and note that $\bar{R}$ is free of rank 2 as an R-module, the above example gives us a projective R-module which induces the $Q \times Q$-module P_+^2. Explicitly, this corresponds to the idempotent 4×4 matrix over C:

$$A = \frac{1}{2}\begin{bmatrix} 1+Z & 0 & X & -Y \\ 0 & 1+Z & Y & X \\ X & Y & 1-Z & 0 \\ -Y & X & 0 & 1-Z \end{bmatrix}.$$

The complementary matrix $I - A$ gives a projective R-module inducing P_-^2.

Combining this example with the observation that the free R-module of rank 1 induces $P_+ \oplus P_-$, and the restrictive results of the preceding section, we have:

PROPOSITION 1. *A $Q \times Q$-module $P_+^a \oplus P_-^b$ is induced by a finitely generated projective R-module (under the identification $Q \cong Q \times R/(ReR)$ if and only if $a \equiv b$ (mod 2).* ∎

We remark that in the results of the preceding sections, we could have replaced Q and $Q(i)$ by $\mathbf{R}$ and $\mathbf{C}$, and polynomial functions by continuous functions, and obtained exactly the same results.

6. *More on the complex case.* The above results on $R \otimes Q(i)$ seem to suggest that the relation between rings R and R/ReR is better if R is an algebra over an algebraically closed field, than in general. But in fact our results on non-induced projectives can be restored in the complex case by replacing S^2 by S^3. I shall only outline the development.

Let $\mathbf{C}$ be the complex numbers, and $C = \mathbf{C}[W, X, Y, Z]/(W^2+X^2+Y^2+Z^2-1)$ the ring of polynomial functions on the unit 3-sphere. We take the option of regarding C as a ring of (complex-valued) functions on the *real* 3-sphere, for this is Zariski-dense in the complex 3-sphere. Defining $f^\rho(W, X, Y, Z) = f(-W, -X, -Y, Z)$, we construct R from C and show $R/ReR \cong \mathbf{C} \times \mathbf{C}$ as before.

An idempotent element $A \in M_n(R)$ now induces a continuous map from S^3 to $G^n(W)$ satisfying (7), but with W now denoting $\mathbf{C}^{2n}$, and G the complex Grassmannian.

Suppose such an idempotent A takes the poles of S^3 to points of $G^a(U) \times G^{m-a}(V)$ and $G^b(U) \times G^{m-b}(V)$. Then if we look along any path $\phi(t)$ ($t \in [0,1]$) joining these poles, say a great semicircle, we get a path ψ in $G^m(W)$. Since complex Grassmannians are simply connected, we may assume up to homotopy that ψ has the form (8). Then (8′) gives a path ψ' which describes the behavior of A on the opposite great semicircle.

We know that ψ and ψ' are homotopic; but since $\pi_2(G^m(W)) = \mathbf{Z}$, there is an infinite family of nonhomotopic homotopies between them. Let one of these be $\psi(t,\theta)$ ($0 \leqq \theta \leqq \pi$, where $\psi(t,0) = \psi(t)$ and $\psi(t,\pi) = \psi'(t)$.) Up to homotopy, this will have one of the following forms:

(9)

$$\psi(t,\theta) = \text{subspace of } W \text{ spanned by } \begin{cases} (1-t)p_j + e^{n(j)i\theta}\, tq_j & (j = 1,\cdots, b-a) \\ p_j & (j = b-a+1,\cdots,b) \\ q_j & (j = b-a+1,\cdots,m-a) \end{cases}$$

where $n(j)$ ($j = 1,\cdots,b-a$) are arbitrary *odd* integers (refer to (8′)). In fact, the homotopy class of $\psi(-,-)$ depends only on their sum.

$\psi(-,-)$ determines L on a great 2-hemisphere of S^3, and again the formula $L(s^\rho) = L(s)'$ extends it uniquely to a great 2-sphere. The extended map is described by the same formula (9), but with $0 \leqq \theta \leqq 2\pi$. Now L will be extendable to a great 3-hemisphere (and hence, using ( )′, to all of S^3) if and only if this map on the 2-sphere is contractible in our Grassmannian. One finds that this is so if and only if $\sum_1^{b-a} n(j) = 0$; to prove this one lifts (9), not to the universal covering space, but to the universal 2-connected fiber space over $G^m(W)$, which consists of all reducible elements of $\Lambda^m W$. But since all $n(j)$ must be odd, their sum can be zero if and only if $b-a$ is even. One deduces as before that $b-a$ even is a necessary condition for $P^a_+ \oplus P^a_-$ to be R-induced. I have not investigated, in this case, whether this condition is also sufficient.

7. *Further questions.* Can our results on non-liftability to finitely generated projective R-modules be strengthened to delete the restriction "finitely generated"?

If R is a *right hereditary* ring, can a factor-ring R/T_P have non-induced finitely generated projectives? What are the global dimensions of the rings R used above?

Will some version of the above examples work in finite characteristic?

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